Solutions to Exercise: MirrorCharge

PF@SuS@UniHD 2016+

In[15]:=
```
SetOptions[Plot, {Frame → True, Filling → Axis,
    ImageSize → 500, PlotStyle → {Thick, Blue}, PlotLegends → None}];
```

In[16]:=
```
Clear[x, y];
```

We calculate the signal induced on a strip of with a by c charge in distance D

Setup

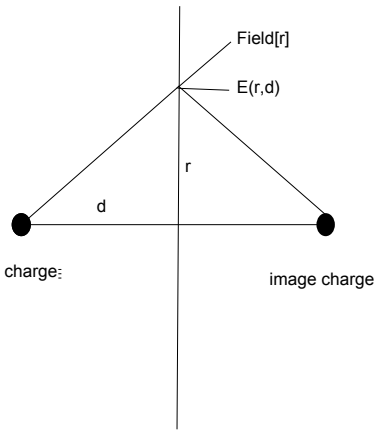
In[17]:=
```
$Assumptions = {d > 0, D > 0, r > 0, x ∈ Reals, a > 0, offset ∈ Reals};
```

1. Surface Charge

In[18]:=
```
Field[r_] = 1/(4 π) 1/r^2; (* Electrical field
    (scalar strength) of ONE unit charge in radial distance r *)
```

In[19]:=
```
∫_0^(4 π) Field[r] r^2 dΩ == 1  (* check that Gauss's Law is fulfilled *)
```

Out[19]=
```
True
```



In[20]:=
```
Qr[r_, D_] = 2 D/Sqrt[r^2 + D^2] Field[Sqrt[r^2 + D^2]] (* Field plane,
    perpenticlular. Factor 2 is from mirror charge *)
```

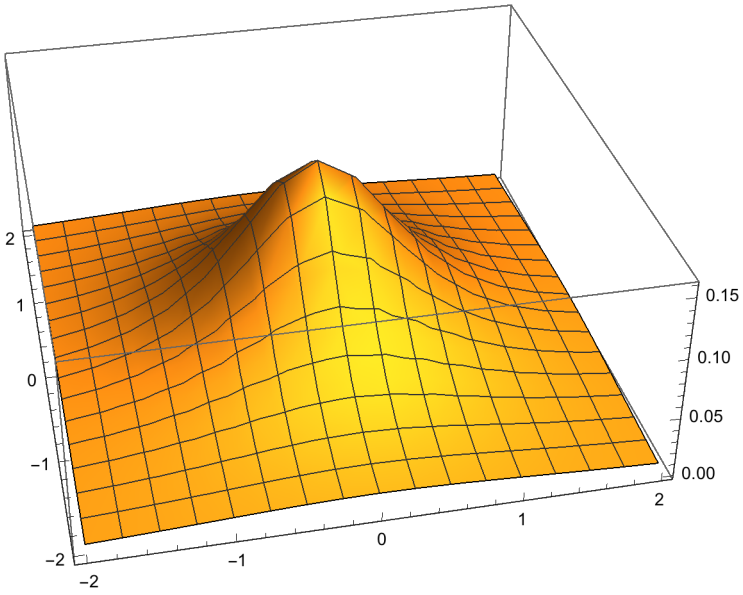
Out[20]=
$$\frac{D}{2\,\pi\,\left(D^2+r^2\right)^{3/2}}$$

In[21]:=
```
Qxy[x_, y_, D_] = Qr[Sqrt[x^2 + y^2], D]
  (* Also express this as a function of x and y instead of r *)
```

Out[21]=
$$\frac{D}{2\pi\left(D^2+x^2+y^2\right)^{3/2}}$$

In[22]:=
```
Plot3D[Qxy[x, y, 1], {x, -2, 2}, {y, -2, 2}]
```

Out[22]=



2. Check Total Charge

In[23]:=
```
∫_0^(2 π) ∫_0^∞ Qr[r, d] r dr dφ == 1
  (* Check that integral over plane is ok: radial case *)
```

Out[23]=
```
True
```

In[24]:=
```
∫_-∞^∞ ∫_-∞^∞ Qxy[x, y, d] dx dy == 1 (* Check that integral over plane is still ok *)
```

Out[24]=
```
True
```

3. Strip Charge

In[25]:=
```
Qstrip[x_, D_] =
  ∫_-∞^∞ Qxy[x, y, D] dy  (* charge DENSITY along an infinite strip in y *)
```

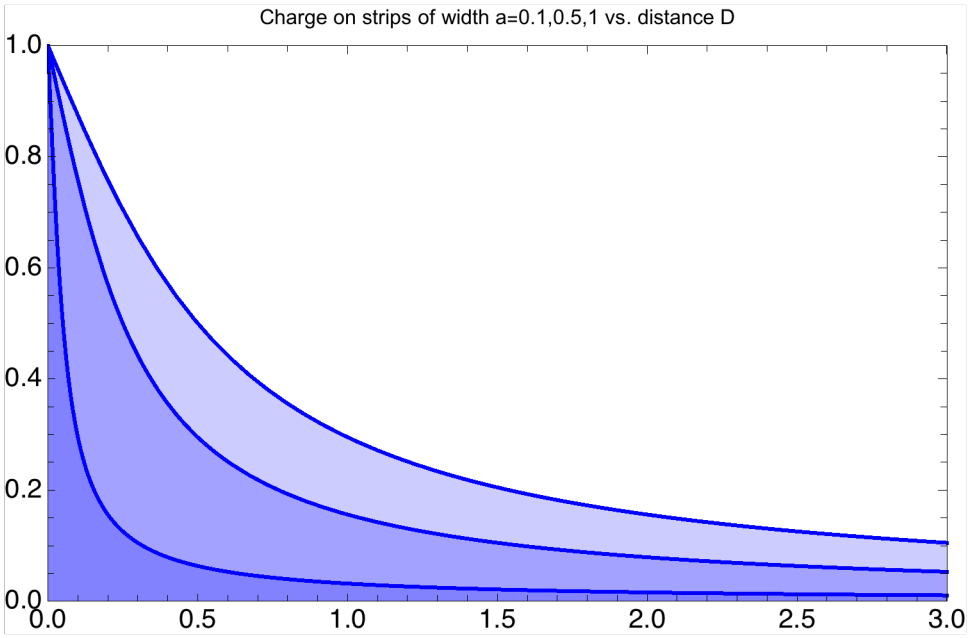
Out[25]=
$$\frac{D}{\pi\left(D^2+x^2\right)}$$

In[26]:=
$$Qstrip[D\_] = \int_{-a/2}^{a/2} Qstrip[x, D]\, dx$$

Out[26]=
$$\frac{2\,\text{ArcTan}\left[\frac{a}{2\,D}\right]}{\pi}$$

In[27]:=
```
Plot[Qstrip[D] /. a → {0.1, 0.5, 1}, {D, 0, 3}, PlotRange → {0, 1}
 , PlotLabel → "Charge on strips of width a=0.1,0.5,1 vs. distance D"]
(* All charge on strip for d=0 more charge on wider strips *)
```

Out[27]=



In[30]:=
```
Qstrip[D] /. a → 1
```

Out[30]=
$$\frac{2\,\text{ArcTan}\left[\frac{1}{2\,D}\right]}{\pi}$$

In[36]:=
```
Limit[Qstrip[D], D → 0, Direction → "FromAbove"] == 1
 (* All signal is induced on strip if charge is very close *)
```

Out[36]=
```
True
```

4. Special case

In[37]:=
$$Qstrip\left[\left\{a, \frac{a}{2}\right\}\right] \text{ // Simplify}$$

Out[37]=
$$\left\{\frac{2\,\text{ArcTan}\left[\frac{1}{2}\right]}{\pi}, \frac{1}{2}\right\}$$

In[38]:=
```
% // N
```

Out[38]=
```
{0.295167, 0.5}
```

5. Signal with offset

In[39]:=
```
Clear[Signal] (* In case this is already defined*)
```

In[40]:=
```
Signal[offset_, a_, D_] =
 ∫_(offset-a/2)^(offset+a/2) Qstrip[x, D] dx  (* Signal on strip of width a at offset *)
```

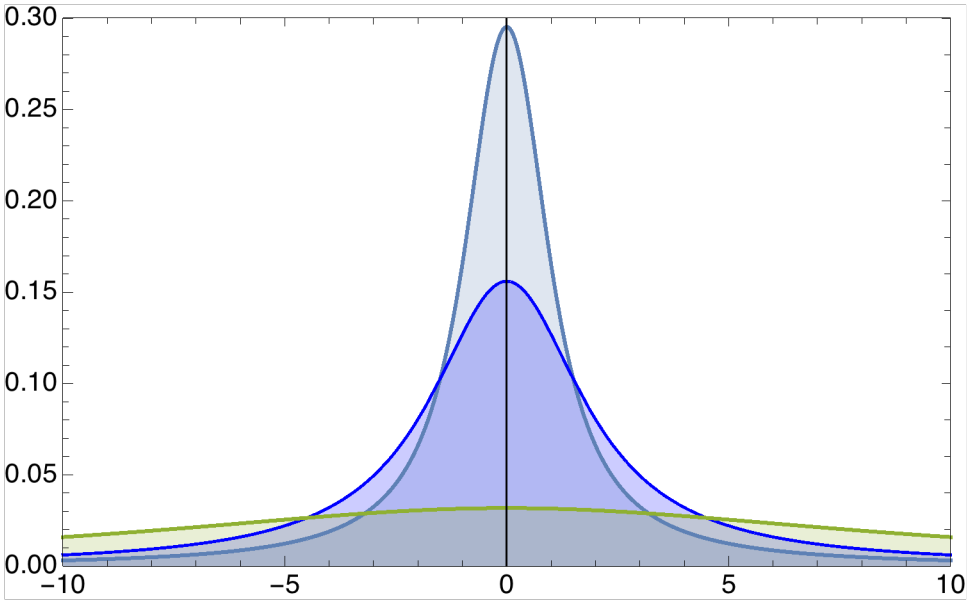
Out[40]=
$$\frac{\text{ArcCot}\left[\frac{2\,D}{a-2\,\text{offset}}\right]+\text{ArcCot}\left[\frac{2\,D}{a+2\,\text{offset}}\right]}{\pi}$$

In[41]:=
```
Signal[0, a, D] == Qstrip[D] // FullSimplify
 (* Check special case offset=0 with old result *)
```

Out[41]=
```
True
```

In[42]:=
```
Plot[Signal[os, 1, {1, 2, 10}] // Evaluate,
 {os, -10, 10}, Exclusions → 0, PlotRange → {0, 0.3}]
```

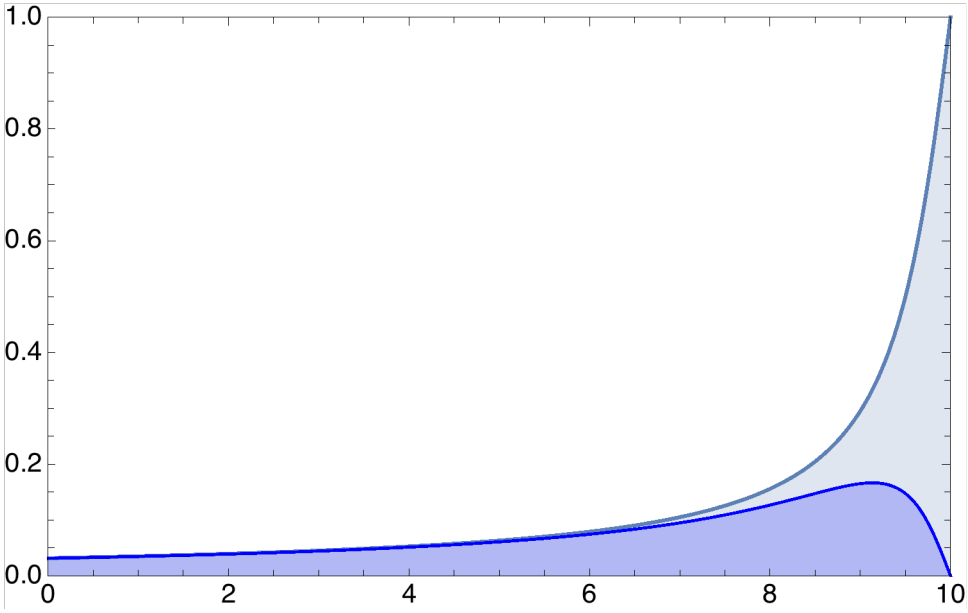
Out[42]=



6. Now drift charge towards strip

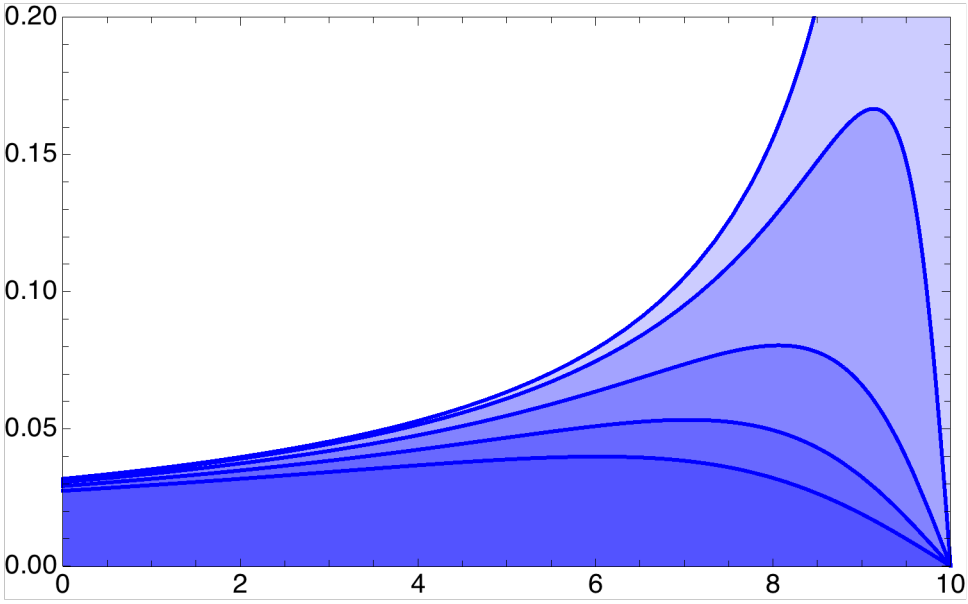
In[43]:=
```
Plot[{Signal[0, 1, 10 - d], Signal[1, 1, 10 - d]}, {d, 0, 10}, PlotRange → {0, 1}]
(* Signal on central strip and on neighbor *)
```

Out[43]=



In[44]:=
```
Plot[Table[Signal[offset, 1, 10 - d], {offset, 0, 4, 1}], {d, 0, 10},
 PlotRange → {0, 0.2}]  (* Signal on central strip and on neighbor *)
```

Out[44]=



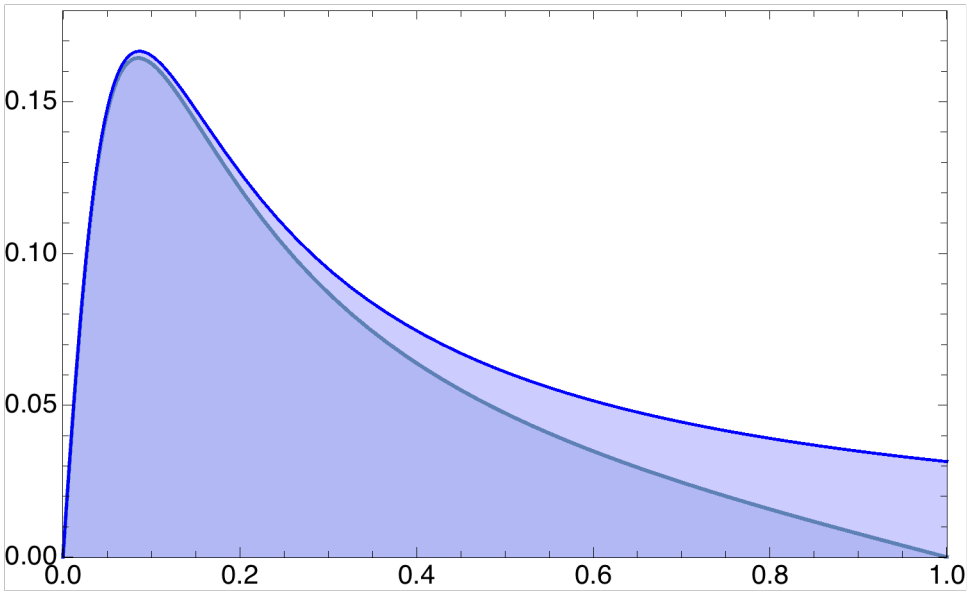
Compare with exact weighting field solution

In[45]:=
```
MyArcTan[x_] := If[x > 0, ArcTan[x], ArcTan[x] + π];
```

In[46]:=
```
Phi[x_, y_, a_] := 1/π MyArcTan[Sin[π y] Sinh[Pi a / 2] / (Cosh[Pi x] - Cos[Pi y] Cosh[Pi a / 2])];
```

In[47]:=
```
Plot[{Phi[0.1, d, 0.1], Signal[0.1, 0.1, d]},
 {d, 0, 1}, Frame → True, PlotRange → {0, 0.18}]
```

Out[47]=



The difference is that the weighting potential is for a grounded backplane!